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Axiomatic and Game Theoretic Models of Bargaining

There is no such thing as Islamic science, just as there is no Hindu science, no Jewish science, no Confucian science...
Abdus Salam

15.1 Introduction

A *bargaining model* is a game in which two or more players stand to gain by cooperating, but they must negotiate an acceptable procedure for sharing the gains from cooperation. The problem in modelling bargaining is to determine whether and when agreement will be reached, and how the gains will be divided, depending on the bargaining rules and the characteristics of the bargainers. We have encountered several bargaining models already—the Ståhl-Rubinstein bargaining model in (§5.6), a finite bargaining game solvable by backward induction in (§5.7), and an incomplete information bargaining game with a Pareto-inefficient outcome in (§12.3).

Game theory has made especially important contributions in the area of bargaining. As the following models illustrate, there are several plausible implications of analytical bargaining models, including the following:

(a) Where all parties have complete knowledge, there are generally Pareto efficient bargaining equilibria, in which there are no impasses, strikes, or wars, and all divorces are amicable;

(b) A bargainer with an improved fallback position (the payoff if agreement cannot be reached) receives a larger payoff. Thus if there is near full employment, workers do better in bargaining with employers, even though begin dismissed occurs with zero probability in equilibrium. Similarly a country with an imposing standing army does better in negotiations, even though the probability of war is zero.

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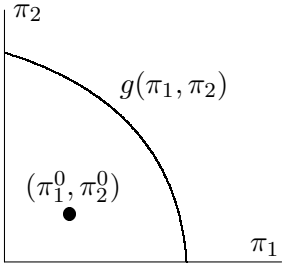
(c) A bargainer who becomes less risk averse receives a larger payoff. Since the wealthy tend to be the least risk averse, because utility functions tend to display decreasing absolute risk aversion (§rbsufsec), the relatively wealthy bargainer does better;

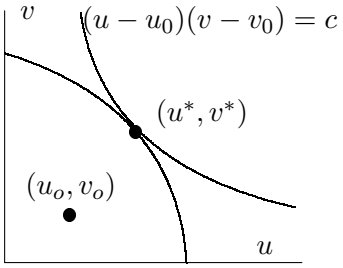
(d) A bargainer who discounts the future less heavily receives a higher payoff. Thus workers with a large strike fund can bargain more effectively, and a person who needs a plumber in a hurry to fix a leak is in a weak bargaining position.

There is of course much more to learn about bargaining. For one thing, when there is incomplete knowledge outcomes are generally not Pareto efficient, and may be strongly dependent upon details of the bargaining process (Kennan and Wilson 1990). Moreover, we have stressed that people rarely choose strategies by relying on many levels of backward induction, and they (correctly) assume the players they face are similarly disinclined (§2.18). Where evolutionary pressures are brought to bear on particular bargaining situations, and if mutation and error rates are sufficiently low, we might expect subgame perfect equilibria to dominate in the long run. But there is no reason to believe these conditions hold in the real world bargaining situations we attempt to model. Moreover, contrary to standard bargaining models, people care about the *process* of bargaining as well the *outcome*, much as they do to dictator, ultimatum, and public goods games (see Chapter 11), where issues of fairness and the intentions of other players strongly affect behavior. We need models that reflect these realities.

15.2 The Nash Bargaining Model



Two players bargain over a division (π_1, π_2), where π_1 and π_2 satisfy the differentiable relationship $g(\pi_1, \pi_2) = 0$. Suppose the two players have utility functions $u(\pi_1)$ and $v(\pi_2)$, respectively. If the two players cannot come to an agreement, the players have fallback payoffs π_1^o and π_2^o, respectively. For instance, the players may be splitting an amount of money M, so $\pi_1 + \pi_2 = 0$, and $\pi_1^o = \pi_2^o = 0$, since if they cannot agree on a split they each get nothing. We assume the curve defined by $g(\pi_1, \pi_2)$ is downward sloping and convex, as in the diagram to the right.



John Nash showed that under certain (sometimes plausible) conditions, the two will agree to a pair (π_1^*, π_2^*) that maximizes the expression $(u(\pi_1)-u(\pi_1^o))(v(\pi_2)-v(\pi_2^o))$ subject to the constraint $g(\pi_1, \pi_2) = 0$. This is called the *Nash bargaining solution.* This solution is illustrated in the figure to the right, where we have moved from (π_1, π_2) axes to (u, v) axes, so the Nash bargaining solution is the point $(u^*, v^*) = (u(\pi_1^*), v(\pi_2^*))$ where the Pareto frontier $\{(u(\pi_1), v(\pi_2))|g(\pi_1, \pi_2) = 0\}$ is tangent to the isoquant $(u - u^o)(v - v^o)$. Note that in utility space, the fallback becomes $(u^o, v^o) = (u(\pi_1^0), v(\pi_2^o))$. Here is a rigorous statement of Nash's theorem.

A *bargaining problem* (S, d_1, d_2) consists of a set $S = \{(x, y)|x, y \in \mathbf{R}\}$ of ordered pairs of feasible payoffs to two agents, and a pair of points $(d_1, d_2) \in S$ representing the agents' fallback payoffs if they cannot agree. Agreement consists of choosing a point $(\pi_1, \pi_2) \in S$ so that the first agent gets π_1 and the second gets π_2.

A *bargaining solution* is a pair of functions $\pi_1 = f_1(S, d_1, d_2)$ and $\pi_2 = f_2(S, d_1, d_2)$ that assign to a bargaining problem (S, d_1, d_2) a pair of payoffs (π_1, π_2), one to each of the two agents. Suppose a bargaining problem has the following four properties:

(a) **Invariance to Equivalent Utility Representations:** If we replace $u(\pi_1)$ by $\alpha u(\pi_1) + \beta$ or $v(\pi_2)$ by $\alpha v(\pi_2) + \beta$, for $\alpha > 0$, the solution does not change.

(b) **Pareto Efficiency:** Suppose $(f_1(S, d_1, d_2), f_2(S, d_1, d_2)) = (\pi_1, \pi_2) \in S$, and let $(t_1, t_2) \in S$. Then if $u(t_1) > u(\pi_1)$, we must have $v(t_2) < v(\pi_2)$.

(c) **Symmetry:** If $(\pi_2, \pi_1) \in S$ whenever $(\pi_1, \pi_2) \in S$, if $d_1 = d_2$, and if $u(\pi) = v(\pi)$ for all $s \in S$, then $f_1(\pi_1, \pi_2, d_1, d_2) = f_2(\pi_1, \pi_2, d_1, d_2)$.

(d) **Independence from Irrelevant Alternatives:** If (S, d_1, d_2) and (T, d_1, d_2) are bargaining problems with $S \subseteq T$ and

$$(f_1(T, d_1, d_2), f_2(T, d_1, d_2)) \in S,$$

then $f_1(S, d_1, d_2) = f_1(T, d_1, d_2)$ and $f_2(S, d_1, d_2) = f_2(T, d_1, d_2)$;

The *Invariance* property says that the outcome cannot depend on the arbitrary units in which we measure utility, and *Pareto efficiency* says

that any bargaining solution exhausts all opportunities for making both players better off. Critics often claim that Pareto efficiency is unrealistic, since bargaining breaks down all the time in the real world (strikes, wars, bitter divorces, expensive litigation). However, bargaining takes place extremely frequently in social and economic life, and only rarely does it break down. Breakdowns, though, are highly salient events, and the inability of the Nash bargaining model to portray bargaining breakdown, that can be addressed by developing game theoretic bargaining models with asymmetric information.

The *Symmetry* assumption says that neither player is preferred in any way to the other, so players are distinguished only by their utility functions and fallback positions. *Independence from Irrelevant Alternatives* is the most subtle assumption. This property says that the outcome of a bargain cannot depend on the availability of alternative bargains that are not in fact chosen when players have the option to do so. One common form of asymmetry in bargaining is the fact that in alternating offer bargaining models (§5.6, §5.6, and §5.7) the player who goes first often has an advantage.

Theorem 15.1. Nash's Bargaining Theorem*: Nash's Bargaining Theorem There is a unique bargaining solution*

$$(\pi_1, \pi_2) = (f_1(S, d_1, d_2), f_2(S, d_1, d_2))$$

that satisfies the properties (a)–(d). This solution defines (π_1, π_2) *as the payoff pair* (s_1, s_2) *that maximizes* $(u(s_1) - u(d_1))(v(s_2) - v(d_2))$ *subject to the constraint* $(s_1, s_2) \in S$*.*

This solution is called the *Nash bargaining solution.* The proof of Nash's Bargaining Theorem is not difficult—see Osborne and Rubinstein (1990) for details. Discovering the theorem, on the other hand, is a stroke of genius.

Notice that the Nash bargaining model is neither a normal nor an extensive form game. The question then arises as to whether there are plausible games that satisfy the Nash axioms. We explore this issue below.

(a) Prove that in a Nash bargaining situation, a player whose fallback position improves gains at the expense of the other player.

(b) Show that this fact remains true when we extend the Nash bargaining solution to $n > 2$ players. In this case, $\pi_1^*, \ldots, \pi_n^*$ maximizes

$$\max_{\pi_1,\ldots,\pi_n} \prod_{i=1}^{n} (u_i(\pi_i) - u_i(\pi_i^o))$$

subject to the constraint $g(\pi_1, \ldots, \pi_n) = 0$.

15.3 Risk Aversion and the Nash Bargaining Solution

In the Nash bargaining model, the more risk averse (§rbsufsec) an agent is, the worse the agent does in bargaining. To see this, suppose the Pareto frontier of the bargaining set is given by $y = g(x)$, where $g(x)$ is decreasing and concave, the agents have concave utility functions $u_1(x)$ and $u_2(y)$, and the fallback is $u_1 = u_2 = 0$. Maximizing $u_1(x)u_2(y)$ gives $u_1'(x)/u_1(x) = -u_2'(y)y'/u_2(y)$. The curve $u_1'(x)/u_1(x)$ is downward-sloping, since its derivative is $(u_1''u_1 - (u')^2)/u_1^2 < 0$. Similarly, the curve $u_2'(y)y'/u_2(y)$ is upward-sloping, since its derivative is $-(u_2''(y)(y')^2 + u_2'y'')u_2'(y) - u_2'(y)^2(y')^2/u_2(y)^2$, which is positive. This gives the Nash bargaining equilibrium at z^* in Figure 15.1.

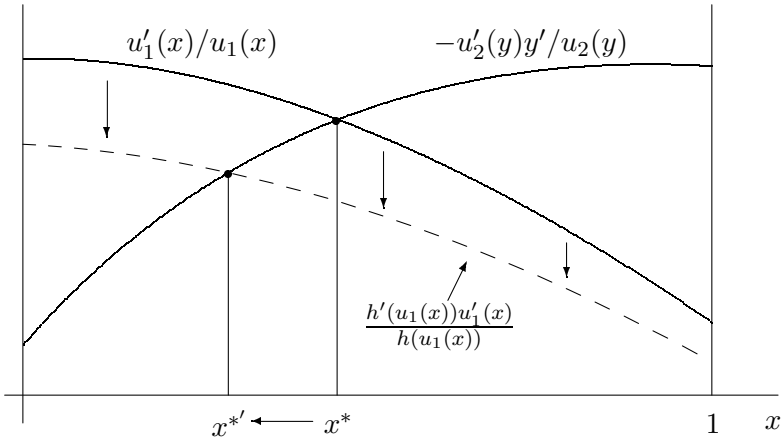


Figure 15.1: Bargaining with Risk Averse Agents

Now suppose the first agent becomes more risk averse. What does this mean for the agent's utility function? Specifically, suppose the original utility function $u_1(x)$ with $u_1(0) = 0$ becomes the more risk averse utility function $v_1(x)$ with $v_1(0) = 0$.

The Arrow-Pratt measure of risk aversion is

$$\lambda_u(x) = -\frac{u''(x)}{u'(x)},$$

for utility function $u(x)$. The higher is $\lambda_u(x)$, the greater the risk premium the agent must receive to be willing to accept a small lottery around x, rather than have the expected value of the lottery. We say $v(x)$ is *uniformly more risk averse* than $u(x)$ if $\lambda_v(x) > \lambda_u(x)$ for all x (§rbsufsec).

Utility function $v(x)$ with $v(0) = 0$ is uniformly more risk averse than utility function $u(x)$ with $u(0) = 0$ if and only if there is a strictly concave function $h(\cdot)$ with $v(x) = h(u(x))$ for all x and $h(0) = 0$. Using this fact, we can write the agent's new utility function as $v_1(x) = h(u_1(x))$, where $h : \mathbf{R} \to \mathbf{R}$ is increasing and concave. Now the curve $u_1'(x)/u_1(x)$ is replaced by $h'(u_1(x))u_1'(x)/h(u_1(x))$ which is the former curve shifted downward. To see this, we need only show that $h'(t) < h(t)/t$. This is obvious geometrically (see Figure 15.2), but using calculus, we see that $th'(t) = h(t)$ when $t = 0$, and $(th')' = h' + th'' < h'$, so $th'(t)$ increases more slowly than $h(t)$.

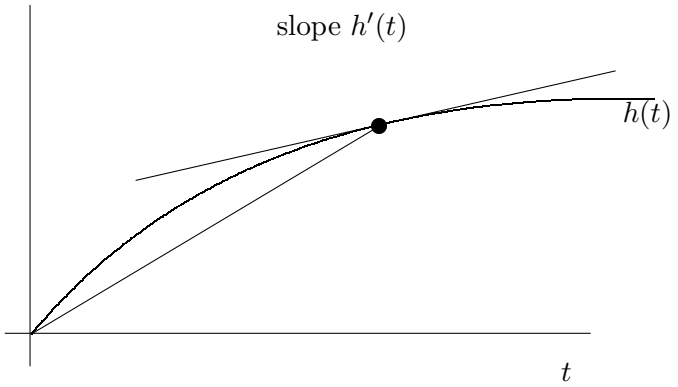


Figure 15.2: Increasing Risk Aversion

Since the curve $u_1'(x)/u_1(x)$ shifts down when agent 1 becomes more risk averse, her share decreases (from x^* to $x^{*'}$ in the Figure 15.2). This proves the assertion.

15.4 The Ståhl-Rubinstein Bargaining Model with Responent Outside Options

Two agents bargain over splitting a dollar using the Ståhl-Rubinstein alternating offer bargaining model with common discount rate δ (§5.6). Suppose that as a respondent, player i can accept or reject the offer. If the offer is rejected, an outside option worth s_i materializes with probability $p > 0$. If the offer materializes, the respondent can accept or reject the outside offer. If the outside offer is accepted, the respondent's payoff is s_i and the proposer's payoff is zero. Otherwise the proposer and the respondent exchange roles and the game continues. The game tree is depicted in Figure 15.3.

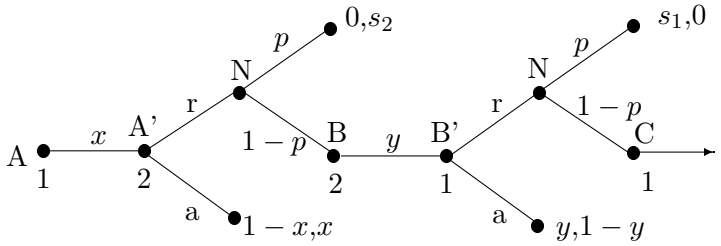


Figure 15.3: Ståhl-Rubinstein Bargaining with Outside Options

The following exercise derives a subgame perfect Nash equilibrium of this bargaining game. Using the technique of §5.6, you can show that this is the *unique* subgame perfect Nash equilibrium. Use the One Stage Deviation Principle (§6.5) to do the following problem.

(a) Show that if $s_i > (1-(1-p)\delta)/p$, player i would do better not to bargain and simply wait for the outside option in each period. We assume in the rest of the problem that the reverse inequality holds.

(b) Suppose $s_1, s_2 < \delta/(1+\delta)$. Show that the equilibrium coincides with the Ståhl-Rubinstein model without outside options: the proposer offers $\delta/(1+\delta)$, and the respondent accepts anything greater than $\delta/(1+\delta)$ while rejecting anything less than this amount. The payoffs are $\pi_1 = 1/(1+\delta)$ and $\pi_2 = \delta/(1+\delta)$.

(c) Let

$$\pi_1 = \frac{1-(1-p)\delta - ps_2}{1-(1-p)\delta^2}$$

and suppose $s_1 < \delta\pi_1$ and $\delta/(1+\delta) < s_2$. Show that there is a

SPNE in which player 1 offers $1 - \pi_1$, player 2 accepts, and the payoffs are $(\pi_1, 1 - \pi_1)$.

(d) Let

$$\pi_1 = \frac{1 - \delta + \delta p s_1}{1 - (1 - p)\delta^2}$$

and suppose $s_2 < \delta\pi_1$ and $\delta/(1 + \delta) < s_1$. Show that there is a SPNE in which player 1 offers $1 - \pi_1$, player 2 accepts, and the payoffs are $(\pi_1, 1 - \pi_1)$.

(e) Let

$$\pi_1 = \frac{1 - (1 - p)\delta(1 - ps_1) - ps_2}{1 - (1 - p)^2\delta^2},$$

and suppose $s_1, s_2 > \delta\pi_1$. Then there is an SPNE in which each player offers $1 - \pi_1$ and seeks the outside option if offered less than this amount.

This solution looks complicated, but if we graph it, the result is very straightforward. Figure 15.4 shows the relationship between π_1 and s_2 when $0 < s_1 < \delta/(1 + d) < s_1/\delta$.

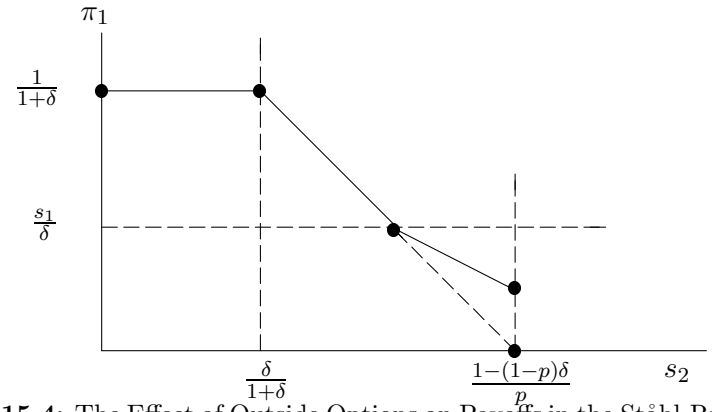


Figure 15.4: The Effect of Outside Options on Payoffs in the Ståhl-Rubinstein Bargaining Model

15.5 Bargaining with Two-Sided Outside Options

The previous problem assumes that the respondent has an outside option available. One would think that the same variety of equilibria would obtain if the proposer had an outside option, but such is not at all the

case (Ponsati and Sákovics 1998). Suppose One and Two bargain for \$1, where One has an outside option x_1 and Two has an outside option x_2, where $0 \leq x_1 \leq 1 - x_2 \leq 1$. If an offer is rejected either player can take the outside option, forcing the other to take the outside option as well. Notice that this includes $x_1 = x_2 = 0$, so the outside option is not available, but the players are not forced to continue bargaining. Show the following:

(a) If One is first to offer, there is a subgame perfect equilibrium in which One offers x_2 and Two accepts. Notice that this implies that if $x_2 = 0$ then One gets the whole dollar!

(b) Suppose One's discount factor is $\delta_1 \in (0, 1)$ and Two's discount factor is $\delta_2 \in (0, 1)$, and suppose $x^* \in [1 - \delta_2(1 - x_1/\delta_1), 1 - x_2]$. Then there is a subgame perfect equilibrium in which One offers Two $1 - x^*$ and Two accepts. Hint: Show that Two will accept an offer on the first round only if (i) Two is indifferent to accepting vs. waiting until the next period; (ii) both players (weakly) prefer the agreement to the outside options; and (iii) One (weakly) prefers continuing to taking the option.

(c) For any period $t > 0$, suppose $x^* \in [(1 - \delta_2(1 - x_1/\delta_1))/\delta_1, (1 - \delta_1(1 - x_2/\delta_2))/\delta_2^{t-1}]$. Then there is a subgame perfect equilibrium in which offers are refused for the first $t - 1$ rounds, and on the t^{th} One receives x^* and Two receives $1 - x^*$.

15.6 How the Ståhl-Rubinstein Bargaining Model Becomes the Nash Bargaining Model

The Nash bargaining model is useful only if there are plausible games that satisfy the Nash bargaining axioms. The Ståhl-Rubinstein model does not accomplish this, because it violates the symmetry axiom—player 1 has a first mover advantage. We could correct this by having the players flip a fair coin to see who goes first, but we still have the problem that the Ståhl-Rubinstein model uses player discount factors rather than utility functions. In this and the following section we explore models that do implement Nash bargaining.

Suppose two agents bargain over the choice of an allocation $s \in S$, where $S \subset \mathbf{R}^2$ is a closed, bounded, convex set. We write the Pareto frontier of this set as $y = g(x)$, and we assume $g'(x)$ exists and is negative. Let δ be the discount factor, and suppose the agents take turns making

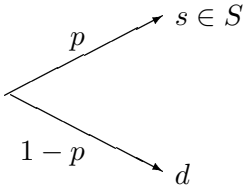
an offer of a pair $(x, y) = (x, g(x))$, until one of the agents accepts the other's offer. We suppose that the value of the payoff declines by a factor of δ per period, and if an offer is rejected, then with a probability $p > 0$ bargaining ends and the agents get their fallback positions (d_1, d_2). We call the subgame perfect Nash equilibrium to this game the Ståhl-Rubinstein bargaining solution for discount factor δ and probability of breakdown p (§5.6).

Show that if the discount factor $\delta = 1$, then as $p \to 0$, the Ståhl-Rubinstein bargaining solution approaches the Nash bargaining solution.

15.7 Zeuthen Lotteries and the Nash Bargaining Solution

The following scenario does implement the Nash bargaining model, and while it is not exactly an extensive form game, it could be made into one with little difficulty—Zeuthen (1930) wrote before game theory was discovered.

Suppose two players are bargaining over the choice of an allocation s in a set S of feasible payoffs. We say an allocation $s \in S$ is "on the table" if one player proposed s. If the other player accepts s, the game is over and the allocation is implemented. If the other player rejects s, then there is a probability $1-p$ that bargaining will break down and the players will get a fallback allocation $d \in S$. Otherwise with probability p, the other agent proposes an alternative $s' \in S$. The game continues until both agree to an allocation, or bargaining breaks down.



We write (p, s) for the *Zeuthen lottery* that pays $s \in S$ with probability p and d with probability $1 - p$ (see the diagram to the right), and we assume agents $i = 1, 2$ have utility functions $u_i : S \to \mathbf{R}$ over the lotteries, and for any two lotteries (p, s) and (p', s') we say . player i prefers the first to the second if $u(p, s) > u(p', s')$, and we say i is indifferent between the two lotteries if $u(p, s) = u(p', s')$. We identify an allocation $s \in S$ that is on the table with the Zeuthen lottery $(1, s)$ that gives s with probability one; i.e., we write s and $(1, s)$ interchangeably. This is because if s is on the table, then the players can obtain s with certainty.

We say an allocation $s^* \in S$ is a *Zeuthen equilibrium* to the bargaining problem if the following condition holds: when s^* is on the table and (p, s) is any Zeuthen lottery, then if one player strictly prefers (p, s) to s^*, the other agent prefers (p, s^*) to s. In other words, if one player

rejects s^* and puts s on the table in the next period, then the other player will reject s and put s^* on the table again. Hence the respondent cannot gain by rejecting s^*.

Notice that Zeuthen bargaining is very close to the Ståhl-Rubinstein bargaining model, except that it does not specify a first mover, and hence treats the players symmetrically. Moreover, a Zeuthen equilibrium is clearly Pareto optimal and is independent of equivalent utility representations. Perhaps then it is not surprising that if $s^* \in S$ is a Zeuthen equilibrium, then s^* is an equilibrium to the Nash bargaining problem; i.e., s^* maximizes

$$(u_1(s) - u_1(d))(u_2(s) - u_2(d)) \qquad s \in S.$$

Show that this is the case.

15.8 Bargaining with Fixed Costs

Suppose Mutt and Jeff bargain over splitting \$1 using alternating offers, but where it costs Mutt c_1 and Jeff c_2 per period if an offer is rejected. Suppose $c_1 < c_2 < c_1 + c_2 < 1$. We will show that there is a unique subgame perfect Nash equilibrium in which, if Mutt goes first, he offers Jeff nothing and Jeff accepts (as usual we assume the *nice guy principle* that a player only rejects an offer if doing so strictly increases the player's expected payoff), and if Jeff goes first, he offers Mutt $1 - c_2$ and Mutt accepts.

(a) Use the One Stage Deviation Principle (§6.5) to show that Mutt's strategy of offering Jeff nothing and Jeff accepting is a subgame perfect Nash equilibrium.

(b) Let m_1 be the most Mutt can get in any subgame perfect Nash equilibrium. Suppose $c_1 \leq m_1 < 1$. Show that this leads to a contradiction.

(c) Suppose $0 < m_1 < c_1$. Show that this leads to a contradiction.

(d) It follows that $m_1 = 1$, so clearly if Jeff goes first, the best he can do is to offer Mutt $1 - c_1$, and keep c_1.

Of course this solution, while mathematically rigorous, is intuitively bizarre, since it predicts an extremely one-sided division no matter how small the costs are, and no matter how small the difference in costs between the two players. Once again backward induction leads to an absurdity.

15.9 Bargaining with Incomplete Information

Suppose a seller has a perishable good whose value to a prospective buyer is v, which is distributed uniformly on an interval $[0, v_1]$. The seller has two chances to make a deal with the buyer, but if the first offer is rejected, the payoffs to both parties after the second offer must be discounted by $\delta < 1$ (§12.3). The seller's strategy set thus consists of price pairs (p_1, p_2), where p_1 is the first period offer, and p_2 is the second period offer, assuming the first period offer is rejected. The buyer's strategies consist accepting and/or rejecting various prices in the two periods. We consider only subgame perfect equilibria, and since the buyer has all the information available to the seller, the buyer also knows the seller's optimal (possibly mixed) strategy. Finally, we assume a buyer will accept an offer unless the buyer can do strictly better by rejecting it (this is the *nice guy principle*).

The follow question asks you to show that the seller will set $p_1 = v_1(2-\delta)/4$ and $p_2 = v_1/4$. Thus once again we find that asymmetry leads to inefficient bargaining, since if $v < v_1/4$, no bargain will be struck between the two.

(a) Suppose the seller knows that if the first offer is rejected, then the value to the prospective buyer is uniformly distributed on $[0, v_2]$. Show that it is optimal for the buyer to set $p_2 = v_2/2$.

(b) Show that if p_1 is rejected, then either $v < p_1, p_2$ or

$$v < \frac{p_1 - \delta p_2}{1-\delta}. \tag{15.1}$$

(c) Show that either $v < p_1, p_2$, or $v_2 = 2p_1/(2-\delta)$ and hence $p_2 = p_1/(2-\delta)$, so the buyer's payoff if the second round is $\delta(v - p_1/(2-\delta))$.

(d) Show that either $v < p_1, p_2$ or $p_1 = v_1(2-\delta)/4$ and $p_2 = v_1/4$.

It follows that the sellers sets $p_1 = v_1(2-\delta)/4$ and $p_2 = v_1/4$. If $v \geq v_1/4$, the first bid is accepted with probability $1/2$; if the first bid is rejected, then v is uniformly distributed on $[0, v_1/2]$, and the second bid is also accepted with probability $1/2$. If $v < v_1(2-\delta)/4$, no deal is made. Note that if the buyer is very impatient ($\delta = 0$), $p_1 = v_1/2$, whereas if the buyer is very patient ($\delta = 1$), $p_1 = v_1/4$. Patient buyers do better.